The third case is an interaction of a shock wave ($M=4$) with an area enlargement (area ratio 0.025). The RCM solution (Fig. 5) shows a transmitted shock wave of a reduced strength and a stationary shock wave within the area-change segment. Due to this intermediate shock, there is also a transmitted contact discontinuity moving about midway between the area change and the transmitted shock wave. This contact discontinuity is not shown on the pressure or velocity distributions (it would show up on a density distribution chart). The GRP solution (Fig. 6) shows essentially the same results, except for a remarkably smoother velocity distribution. The RCM grid had 720 points, while the GRP had just 180 points.

Conclusion

The GRP scheme is superior to the RCM for solving the interaction of rarefaction or shock waves interaction with an area change in a duct. It yields practically noise-free results using a much coarser grid than a similar RCM solution.

Acknowledgment

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References

[1]Schultz-Grunow, F., "Nichtstationaire, Kugelsymmetrische Gasbewegung und Nichtstationaire Gasstromung in Dusen und Diffusoren," *Ingenieur Archiv*, Vol. 14, 1943, pp. 21-29.

[2]Bannister, F. K. and Mucklow, G. F., "Wave Action Following Sudden Release of Compressed Gas from a Cylinder," *Proceedings of Industrial and Mechanical Engineering*, Vol. 159, 1948, pp. 269-300.

[3]Glimm, J., "Solution in the Large for Nonlinear Hyperbolic Systems of Equations," *Communications of Pure and Applied Mathematics*, Vol. 18, 1965, pp. 697-715.

[4]Chorin, A. J., "Random Choice Solution of Hyperbolic Systems," *Journal of Computational Physics*, Vol. 22, 1976, pp. 517-533.

[5]Sod, G. A., "A Numerical Study of a Converging Cylindrical Shock," *Journal of Fluid Mechanics*, Vol. 83, 1977, pp. 785-794.

[6]Greatrix, D. R. and Gottlieb, J. J., "An Analytical and Numerical Study of a Shock Wave Interaction with an Area Change," Institute for Aerospace Studies, University of Toronto, Canada, UTIAS Rept. 268, Nov. 1982.

[7]Gottlieb, J. J. and Igra, O., "Interaction of Rarefaction Waves with Area Reductions in Ducts," *Journal of Fluid Mechanics*, Vol. 137, 1983, pp. 285-305.

[8]Igra, O. and Gottlieb, J. J., "Interaction of Rarefaction Waves with Area Enlargements in Ducts," *AIAA Journal*, Vol. 23, 1985, pp. 1014-1020.

[9]Igra, O., Gottlieb, J. J., and Saito, "An Analytical and Numerical Study of the Interaction of Rarefaction Waves with Area Changes in Ducts. Part 2: Area Enlargements," Institute for Aerospace Studies, University of Toronto, Canada, UTIAS Rept. 282, Aug. 1984.

[10]Ben-Artzi, M. and Falcovitz, J., "A Second-Order Godunov-Type Scheme for Compressible Fluid Dynamics," *Journal of Computational Physics*, Vol. 55, 1984.

[11]Ben Artzi, M. and Falcovitz, J., "An Upwind Second-Order Scheme for Compressible Duct Flows," *SIAM Journal of Scientific and Statistical Computation*, (to be published).

[12]Ben Artzi, M. and Birman, A., "Application of the 'Generalized Riemann Problem' Method to 1-D Compressible Flows with Material Interfaces," University of California, Berkeley, Rept. LBL-18994, 1985.

[13]Ben-Artzi, M. and Falcovitz, J., "A High Order Upwind Scheme for Quasi-One-Dimensional Flows," *Proceedings of the Euler Equations Workshop*, INRIA, Paris, Dec. 1983.

Rocket Motor Flow-Turning Losses

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ANALYSES of axial combustion instabilities in solid propellant rocket motors are based on solutions of the gas phase conservation equations in the rocket combustors. A one-dimensional analysis of the unsteady gas motions is the most basic of these formulations. These one-dimensional approaches encounter difficulties in the attempt to account for the multidimensionality of the flow near the burning propellant surface where the gases have both normal and axial velocity components. This problem has been resolved by Culick[1] by the addition of another term, commonly referred to as "flow turning," into the one-dimensional formulation. This flow-turning term is supposed to account for the effect of the turning of the flow from a direction perpendicular to the chamber boundary to the direction of motion of the longitudinal acoustic waves near the burning propellant surface. Analyses by Culick[1] and Flandro[2] show that this process results in energy losses for the combustor wave motion; these losses are referred to as flow-turning losses.

Two interesting questions, addressed most recently by Hersh and Walter,[3-5] arise in connection with the flow-turning losses. The first concerns the adequacy of a one-dimensional analysis, in which the mass injection from the side walls appears as the wave forcing function, to model the flow-turning losses. Since flow-turning losses involve a directional transfer of momentum, it appears that at least a two-dimensional model of the flowfield would be required, as indicated by Flandro.[2] The second question is whether these losses occur in the vicinity of the chamber walls or in the core flow. If the former is true, then it should be possible to describe the wave motion in the bulk of the chamber by a one-dimensional analysis, provided the flow-turning losses are properly accounted for. In the latter case, however, at least a two-dimensional analysis must be carried out over the entire chamber.

To clarify these statements, consider a constant-area rigid walled duct of high length-to-diameter ratio. Lateral injection of flow is therefore absent. Even in such a case, the longitudinal wave motions are not strictly one-dimensional due to viscous and thermal conduction effects near the wall. However, these effects are important only in a thin layer next to the wall known as the acoustic boundary layer, or Stokes' layer. The extent δ of this boundary layer is given by[6]

$$\delta=\sqrt{2\nu/\omega} \tag{1}$$

where ν is the kinematic viscosity and ω the frequency. (The thermal and viscous boundary layers have approximately the same extent for gases whose Prandtl number is close to unity.) A one-dimensional analysis of the acoustic wave motion in the duct requires the conservation equations to be averaged across the cross section. It may be shown that the relevant wave number k_∞ for longitudinal motions is given by

$$k_\infty^2=\left(\frac{\omega}{c}\right)^2-i2\frac{\omega}{c}\frac{\beta}{a}$$

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where c is the speed of sound, a the relevant cross-sectional dimension, and $\beta=\rho_0 c(v'/p')_{\rm wall}$ the specific acoustic admittance of the side walls, with v' the unsteady normal velocity directed into the chamber. It has been argued by Morse and Ingard[6] that when the acoustic boundary-layer thickness is much smaller than the cross-sectional dimension of the duct, a one-dimensional formulation for the wave motion in the chamber interior is sufficient, provided β is recognized as the admittance at the boundary-layer edge and not at the wall proper. It is to be noted that this admittance at the acoustic boundary-layer edge cannot be obtained by a one-dimensional analysis and at least a two-dimensional analysis of the boundary-layer region is required.

Consider now the case in which there is mass injection through the side walls of the duct. A two-dimensional analysis including this effect along with viscous and thermal conduction effects yields the following acoustic boundary-layer thickness as shown by the authors[7]

$$\delta=\sqrt{\frac{\nu}{\omega}}\left[\frac{\bar{V}}{2\sqrt{2\nu\omega}}\left(1+\frac{\sqrt{1+16\omega^2\nu^2}}{\bar{V}^4}\right)^{1/2}-\frac{\bar{V}}{2\sqrt{\omega\nu}}\right]^{-1} \quad (2)$$

where $\bar{V}$ is the injection velocity. It may be verified that in the limit of $\bar{V}\to 0$ ($\omega\neq 0$), Eq. (1) is obtained. In analogy to the no-injection case, it is expected that wave energy losses and flow-turning effects are confined to this boundary layer. Hence, the wave motion in the chamber interior may be obtained from a one-dimensional analysis, provided the admittance at the boundary-layer edge is correctly applied. This admittance has been obtained by Flandro,[2] and an extended form has been derived by the authors.[7] However, as with the no-injection case, it must be verified that the boundary-layer thickness is small compared to the cross-sectional dimension of the duct. It should be noted that due to the injection velocity the boundary-layer thickness is greater than in the no-injection case. If it turns out that for a given injection velocity and frequency, the boundary-layer thickness becomes comparable to the transverse-duct dimensions, then a one-dimensional analysis will not, in general, adequately describe the duct acoustic wave structure. In such a case, at least a two-dimensional analysis will be required.

It should also be noted that the transverse unsteady velocity will be nonzero at the boundary-layer edge, even when the boundary layer is thin and that this, in fact, provides the admittance there. However, this transverse velocity will, in general, be much smaller than the longitudinal acoustic velocity, enabling a one-dimensional analysis in the fluid interior. Also, from symmetry considerations, this transverse velocity will vanish along the duct centerline. This results in a weak radial dependence of the acoustic pressure, as also happens in the no-injection case.[6]

The recent experimental investigation of flow-turning losses by Hersh and Walker[4] confirm these trends. They conducted experiments in a 7.5×6.25 cm^2 duct with acoustic excitation and uniform injection of flow through the side walls of the test section. They measured the axial acoustic pressure variation along the centerline as well as the acoustic power upstream W_u and downstream W_d of the test section. The power loss at the side walls, W_w, was also determined from measurements of the acoustic admittance at the side walls. The measured difference $W_u - W_d$ was then compared with W_w, the rationale being that if $W_u - W_d > W_w$, then flow-turning losses are present.

Using Hersh and Walker's data,[4] some acoustic boundary-layer thicknesses were calculated, using Eq. (2), and these are presented in Table 1. These calculations show that the boundary-layer thickness increases with increasing injection rate and decreasing frequency of driving. On the basis of the ideas expressed above, it is expected that in the third case the flow-turning losses will occur in the main flow region, whereas in the first two cases, the concept of the boundary layer should remain valid.

The data of Hersh and Walker[4] also show that with increasing injection rate at a given frequency, the difference ($W_u - W_d$) becomes greater than W_w. It should be noted that they compute W_w from the acoustic admittance at the wall rather than that at the acoustic boundary-layer edge. In their experiments, they also note that sound refraction occurs in the setup. This is apparently caused by the fact that the downstream (i.e., in the direction of flow) propagating wave dominates the upstream propagating wave. This results in an increase in the pressure level near the wall as compared to the centerline. They also show that if the admittance at the wall is modified by accounting for the increase in pressure level at the wall as compared to that at the centerline, the difference ($W_u - W_d$) becomes approximately equal to W_w. However, this accounting is done, in effect, by multiplying the measured wall admittance by the ratio of the pressure level at the wall and centerline. Therefore, this modified admittance becomes, effectively, the ratio of the unsteady velocity into the chamber and the centerline pressure (i.e., $v'_{\rm wall}/p'_{\rm center}$). Physically, the admittance at the wall should relate the unsteady velocity and the pressure level at the wall [i.e., $(v'/p')_{\rm wall}$] and not at the centerline. Therefore, this modified admittance cannot be considered as the true wall admittance.

On the other hand, if the admittance at the edge of the acoustic boundary layer is used instead of the wall admittance, the difference between the admittance at the boundary-layer edge and the wall admittance is about the same as the amount obtained by "correcting for refraction." This is true, however, only as long as the boundary-layer thickness is small compared with the duct cross-sectional dimension. Sample calculations for the first two cases in Table 1 are shown in Table 2. The calculations are based on the theory developed by Flandro and extended by the authors. The real part of the admittance is calculated as it determines the power loss.

The data of Hersh and Walker also show that for the first case in Tables 1 and 2, the centerline pressure is adequately represented by their one-dimensional model based on the

Table 1 Sample calculations of acoustic boundary-layer thickness

Injection Mach no.	Frequency, Hz	Boundary-layer thickness, cm (calculated)	Duct/boundary-layer thickness[a]
0.0038	1005	0.3	22.9
0.0054	1005	0.81	8.5
0.0082	812	4.4	1.56

[a] Average cross-sectional dimension.

Table 2 Comparison of admittance at boundary-layer edge with admittance corrected for refraction

Injection Mach no.	Frequency, Hz	$\rho c{\rm Real}(v'/p)_{\rm wall}$ $=A$ (from data)	$\rho c{\rm Real}(v'/p)_{\rm b.l.\ edge}$ $=B$ (from theory)	A	B/A correction factor based on refraction[4]
0.0038	1005	−0.0504	−0.0554	1.099	1.09
0.0054	1005	−0.0504	−0.0577	1.145	1.14

wall admittance. Since the admittance at the boundary-layer edge is close to that at the wall for this case, this is not surprising. With increasing injection rate and lower frequency of driving, and the consequent increase in the boundary-layer thickness, the comparison becomes progressively worse as expected.

Another interesting trend shown by the data of Hersh and Walker is that the flow-turning losses deviate from a linear dependence on the injection Mach number as this Mach number is increased. It is precisely under these conditions that the acoustic boundary-layer thickness becomes comparable to the cross-sectional dimensions of the duct. As long as a legitimate acoustic boundary layer is present, the admittance at its edge will vary directly as the injection Mach number, and thus the flow-turning losses will also vary linearly with the injection Mach number. This apparently does not occur when the boundary layer is no longer thin.

Thus, a one-dimensional model for the acoustic wave structure in a duct with side-wall injection is adequate as long as the acoustic boundary-layer thickness is much smaller than the duct cross-sectional dimension and provided that the acoustic admittance at the boundary-layer edge is used. However, to obtain the admittance analytically requires at least a two-dimensional analysis of the boundary-layer region. If the acoustic boundary-layer thickness becomes comparable to the duct cross-sectional dimension, then the acoustic wave structure in the duct will, in general, need at least a two-dimensional treatment to be modeled accurately. The flow-turning losses occur in the acoustic boundary layer so that when it is thin, the losses may be taken to occur close to the side walls. If the acoustic boundary layer is not thin but encompasses a significant portion of the duct, the flow-turning losses are not confined to the near-wall region.

References

[1]Culick, F. E. C., "The Stability of One-Dimensional Motions in a Rocket Motor," *Combustion Science and Technology*, Vol. 7, No. 4, June 1973, pp. 165-175.

[2]Flandro, G. A., "Solid Propellant Acoustic Admittance Corrections," *Journal of Sound and Vibration*, Vol. 36, No. 3, Oct. 1974, pp. 297-312.

[3]Hersh, A. S. and Walker, B., "Experimental Investigation of Rocket Motor Flow Turning Losses," AIAA Paper 83-1267, 1983.

[4]Hersh, A. S. and Walker, B., "Experimental Investigations of Rocket Motor Flow-Turning Losses," Interim Report, AFRDL Rept. TR-84-009, May 1984.

[5]Hersh, A. S., "Experimental Investigations of Rocket Motor Flow Turning Losses," AIAA Paper 84-2257, 1984.

[6]Morse, P. M. and Ingard, K. U., *Theoretical Acoustics*, McGraw-Hill Book Company, New York, 1968, pp. 289-291, 503-514.

[7]Hegde, U. G., Zinn, B. T., and Chen, F. L., "Investigations of the Acoustic Boundary Layer in Porous Walled Ducts with Flow," AIAA Paper 85-0078, 1985.

Stress Analysis of Short Beams

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Introduction

MANY papers have been published on the analysis of short beams. Using Timoshenko's beam theory, the effects of shearing forces and rotatory inertia to the deflections and frequencies of beams were discussed in detail, and Cowper[1] determined the values of shear coefficients k theoretically for various kinds of cross sections.

Taking into account the warping of the section, Levinson[2] introduced the equations of motion, using equilibrium conditions. He indicated that his formula coincides with the one by the Timoshenko's beam theory provided that $k=5/6$ and obtained the deflections and frequencies[3] of the beams.

On the other hand, Murty[4] insisted that, by merely refining the value of shear coefficient, it is not possible to improve the correlation between theory and experiment. By the principle of minimum total energy, he obtained the fundamental equations governing displacements and determined the frequencies and critical loads.[5] But, restrained conditions obtained in the variational calculus are not always satisfied[6] and, strictly speaking, fundamental equations do not satisfy the equilibrium conditions.

Recently, using the results given in the text by Timoshenko and Goodier,[8] Rehfield and Murthy[7] carried out the stress analysis of the beam simply supported at both ends and discussed the effects of transverse shear, nonclassical axial stress and transverse normal strain to the deflections of beams. Later, by his previous method, Murty[9] analyzed the cantilever beam. His results showed that the shearing stresses along the upper and lower edges do not vanish and, moreover, their values become large in the neighborhood of clamped edge.

In the present paper, taking into account the warping of the section, stress analysis is carried out on the short beam subjected to distributed load. Direct stress σ_x in the axial direction is assumed to be in the form of $\Sigma y^i u_i(x)$ and shearing stress τ and transverse direct stress σ_y are determined, using the equilibrium conditions. The fundamental equations governing u_i are introduced by the variational method. What kinds of u_i should be summed is determined by comparing the values of total energy given by each obtained solution.

Fundamental Equations

The case will be considered where the beam simply supported at both ends is subjected to distributed load q. In order to simplify calculus, q is assumed to be constant along the span.

The equilibrium conditions of stresses are expressed as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

where σ_x, σ_y and τ are direct stresses in the axial and transverse directions and shearing stress, respectively. Now, σ_x is assumed to be in the form of

$$\sigma_x = \eta u_1 + \eta^2 u_2 + \eta^m u_m + \eta^u u_n \tag{2}$$

where 2ℓ, $2h$, m, n, and u_i are length and thickness of the beam, odd and even integers and functions with respect to ξ only, and where $\xi = x/\ell$, $\eta = y/h$, and $r = h/\ell$, respectively.

Substituting Eq. (2) into Eq. (1) and integrating with respect to η, τ and σ_y become

$$\tau = -r\left(\frac{\eta^2}{2}u_1' + \frac{\eta^3}{3}u_2' + \frac{\eta^{m+1}}{m+1}u_m' + \frac{\eta^{n+1}}{n+1}u_n'\right) + f(\xi)$$

$$\sigma_y = r^2\left\{\frac{\eta^3}{6}u_1'' + \frac{\eta^4}{12}u_2'' + \frac{\eta^{m+2}}{(m+1)(m+2)}u_m'' + \frac{\eta^{n+2}}{(n+1)(n+2)}u_n''\right\} - r\eta f' + g(\xi) \tag{3}$$

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